Nonlinear parabolic equation model for finite-amplitude sound propagation over porous ground layers

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Outline

1 Introduction
2 NPE model for rigidly-framed porous media
3 Derivation of coupling equations
4 Numerical examples
5 Forchheimer’s nonlinearities
6 Conclusion & perspectives
Context & objectives

Shock waves from explosions can damage structures

Signals prediction at long range

→ Damages estimation

Propagation:

- Long distances
- High amplitudes
- Meteorological effects
- Ground effects

**Nonlinear Parabolic Equation (NPE)**

method well suited
Existing NPE models

Nonlinear Parabolic Equation (NPE)

\[ D_t R = -\partial_x \left[ c_1 R + c_0 \frac{\beta}{2} R^2 \right] - \frac{c_0}{2} \int \partial_z^2 R \, dx \]

with:

\[ R = \frac{\rho'}{\rho_0} \text{ and } \beta = \frac{\gamma + 1}{2} \]

Assumptions:

1. Propagation along a main direction
2. Weak sound speed perturbations
3. Weak nonlinearities (\( p_{\text{max}} \lesssim 10 \text{ kPa} \))
Current models

- Various formulations (cylindrical, spherical, high-angle, …)
- Thermoviscous effects

Missing feature

Ground effects: soft surfaces
NPE is a time-domain formulation

Complex impedance $Z(\omega)$ not usable $\implies$ ground layer included in the computational system

TO DO:

1. Derive a NPE model for porous ground layers
2. Derive two-way coupling equations (ground/air coupling)
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Ground layer characterized by 4 parameters:

1. Structure constant ($\Phi$)
2. Porosity ($\Omega_0$)
3. Flow resistivity ($\sigma_0$)
4. Forchheimer’s nonlinearity parameter ($\xi$)

Impedance equivalence (linear): $Z(\omega) = \sqrt{\frac{\Phi}{\Omega_0^2}} + i\frac{\sigma_0}{\rho_0\Omega_0\omega}$

From Euler equations (modified Zwikker-Kosten model)...

$$\partial_t (\rho_T) + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\Phi \partial_t (\rho_T \mathbf{v}) + \Phi \nabla \cdot (\mathbf{v} \otimes (\rho \mathbf{v})) + \nabla (p_T) + \sigma_0 \Omega_0 (1 + \xi |\mathbf{v}|) \mathbf{v} = 0$$
...to NPE model for porous ground layers

\[
D_t R = -\frac{c_0}{\sqrt{\Phi}} \partial_x \left[ \left( 1 - \sqrt{\Phi} \right) R + \frac{\beta}{2} R^2 \right] \\
- \frac{c_0}{2\sqrt{\Phi}} \int \partial_z^2 R \, dx - \frac{\sigma_0 \Omega_0}{2\Phi \rho_0} \left( 1 + \xi \frac{c_0}{\sqrt{\Phi}} |R| \right) R
\]

Notes

- **Sound speed**: \( c_{\text{ground}} = \frac{c_0}{\sqrt{\Phi}} \)

- **Nonlinear flow resistivity**: \( \sigma = \sigma_0 \left( 1 + \xi |v| \right) \)

- **NPE model suitable for propagation** within soft ground layers
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First-order coupling equations

How to find expressions for unknowns $p_{i,0}^a$ and $p_{i,1}^g$?

\[ \frac{p_{i,j}'}{p_{i,j}} \]

\[ j = 1 \]

\[ j = 0 \]

\[ p_{i,j} \]
First-order coupling equations

How to find expressions for unknowns \( p_{i,0} \) and \( p_{i,1} \)?

1. Start from linearized vertical flow velocity equations:
   - **Air:** \( \rho_0 \partial_t (w_a) = -\partial_z p_a^T \)
   - **Ground:** \( \Phi \rho_0 \partial_t w_g = -\Omega_0 \partial_z p_g^T - \sigma_0 \Omega_0 w_g \)
First-order coupling equations

How to find expressions for unknowns $p_{i,0}^a$ and $p_{i,1}^g$?

1. Start from linearized vertical flow velocity equations:
   
   **Air:** \[ \rho_0 \partial_t (w^a) = -\partial_z p_T^a \]
   
   **Ground:** \[ \Phi \rho_0 \partial_t w^g = -\Omega_0 \partial_z p_T^g - \sigma_0 \Omega_0 w^g \]

2. Apply parabolic approximation, write \[ [w^\text{air}] = [w^\text{ground}] : \]
   
   \[ \left[ \sqrt{\Phi} \partial_z p_{\text{air}}^a - \frac{\sigma_0 \Omega_0}{\rho_0 c_0} \int \partial_z p_{\text{air}}^a \, dx \right] = \left[ \Omega_0 \partial_z p_{\text{ground}}^g \right] \]
First-order coupling equations

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2. Apply parabolic approximation, write $[w^\text{air}] = [w^\text{ground}]$:
   $$\left[ \sqrt{\Phi} \partial_z p_{i,j}^{\text{air}} - \frac{\sigma_0 \Omega_0}{\rho_0 c_0} \int \partial_z p_{i,m}^{\text{air}} \, dx \right] = \left[ \Omega_0 \partial_z p_{i,j}^{\text{ground}} \right]$$

3. Discretization: e.g. 1st order finite differences, trapezoidal law:
First-order coupling equations

How to find expressions for unknowns $p_{i,0}^a$ and $p_{i,1}^g$?

1. Start from linearized vertical flow velocity equations:
   - Air: $\rho_0 \partial_t (w^a) = -\partial_z p_T^a$
   - Ground: $\Phi \rho_0 \partial_t w^g = -\Omega_0 \partial_z p_T^g - \sigma_0 \Omega_0 w^g$

2. Apply parabolic approximation, write $[w^\text{air}] = [w^\text{ground}]$:
   \[
   \left[ \sqrt{\Phi} \partial_z p_{i,j}^a - \frac{\sigma_0 \Omega_0}{\rho_0 c_0} \int \partial_z p_{i,j}^a \, dx \right] = \left[ \Omega_0 \partial_z p_{i,j}^g \right]
   \]

3. Discretization: e.g. 1st order finite differences, trapezoidal law:
   \[
   (A+G)p_{i,0}^{',a} = (A-G)p_{i,1}^{',a} + 2Gp_{i,0}^{',g} + S_A \sum_{m=N_x}^{i+1} (p_{m,1}^{',a} - p_{m,0}^{',a})
   \]
   \[
   (A+G)p_{i,1}^{',g} = (G-A)p_{i,0}^{',g} + 2Ap_{i,1}^{',a} + S_A \sum_{m=N_x}^{i+1} (p_{m,1}^{',a} - p_{m,0}^{',a})
   \]
   \[
   A = \sqrt{\Phi} + \frac{\sigma_0 \Omega_0 \Delta x}{2c_0 \rho_0}, \quad G = \Omega_0, \quad S_A = \frac{\sigma_0 \Omega_0 \Delta x}{c_0 \rho_0}
   \]
Two-way coupling equations:

\[(A + G) p_{i,0}^a = (A - G) p_{i,1}^a + 2G p_{i,0}^g + S_A \sum_{m=N_x}^{i+1} (p_{m,1}^a - p_{m,0}^a)\]

\[(A + G) p_{i,1}^g = (G - A) p_{i,0}^g + 2A p_{i,1}^a + S_A \sum_{m=N_x}^{i+1} (p_{m,1}^a - p_{m,0}^a)\]

\[A = \sqrt{\Phi} + \frac{\sigma_0 \Omega_0 \Delta x}{2c_0 \rho_0}, \quad G = \Omega_0, \quad S_A = \frac{\sigma_0 \Omega_0 \Delta x}{c_0 \rho_0}\]

First-order limitations:

Linearized Euler equations are used: Forchheimer’s nonlinearities cannot be accounted for, but approximate solutions exist...
Two-way coupling equations:

\[
(A + G) p_{i,0}^a = (A - G) p_{i,1}^a + 2G p_{i,0}^g + S_A \sum_{m=N_x}^{i+1} (p_{m,1}^a - p_{m,0}^a)
\]

\[
(A + G) p_{i,1}^g = (G - A) p_{i,0}^g + 2A p_{i,1}^a + S_A \sum_{m=N_x}^{i+1} (p_{m,1}^a - p_{m,0}^a)
\]

\[
A = \sqrt{\Phi} + \frac{\sigma_0 \Omega_0 \Delta x}{2c_0 \rho_0}, \quad G = \Omega_0, \quad S_A = \frac{\sigma_0 \Omega_0 \Delta x}{c_0 \rho_0}
\]

Causality:

Coupling at range \( i \) involves values at range \( N_x \rightarrow i \)
Two-way coupling equations:

\[(A + G)p_{i,0}^a = (A - G)p_{i,1}^a + 2Gp_{i,0}^g + S_A \sum_{m=N_x}^{i+1} (p_{m,1}^a - p_{m,0}^a)\]

\[(A + G)p_{i,1}^g = (G - A)p_{i,0}^g + 2Ap_{i,1}^a + S_A \sum_{m=N_x}^{i+1} (p_{m,1}^a - p_{m,0}^a)\]

\[A = \sqrt{\Phi} + \frac{\sigma_0 \Omega_0 \Delta x}{2c_0 \rho_0}, \ G = \Omega_0, \ S_A = \frac{\sigma_0 \Omega_0 \Delta x}{c_0 \rho_0}\]

Consistency: rigid boundary

- Setting \(\Phi \gg 1\) gives:

\[p_{i,0}^a = p_{i,1}^a \quad \rightarrow \quad \partial_z p^a = 0 \ (1^{st} \text{order FD})\]
Two-way coupling equations:

\[(A + G)p_{i,0}^a = (A - G)p_{i,1}^a + 2Gp_{i,0}^g + S_A \sum_{m=N_x}^{i+1} (p_{m,1}^a - p_{m,0}^a)\]

\[(A + G)p_{i,1}^g = (G - A)p_{i,0}^g + 2Ap_{i,1}^a + S_A \sum_{m=N_x}^{i+1} (p_{m,1}^a - p_{m,0}^a)\]

\[A = \sqrt{\Phi + \frac{\sigma_0 \Omega_0 \Delta x}{2c_0 \rho_0}}, \quad G = \Omega_0, \quad S_A = \frac{\sigma_0 \Omega_0 \Delta x}{c_0 \rho_0}\]

Consistency: transparent boundary

- Setting \(\sigma_0 = 0\), \(\Omega_0 = 1\) and \(\Phi = 1\) gives:

\[p_{i,0}^a = p_{i,0}^g \text{ and } p_{i,1}^g = p_{i,1}^a\]
Two-way coupling equations:

\[
(A + G) p'_{i,0} = (A - G) p'_{i,1} + 2Gp'_{i,0} + S_A \sum_{m=N_x}^{i+1} (p'_{m,1} - p'_{m,0})
\]

\[
(A + G) p'_{i,1} = (G - A) p'_{i,0} + 2Ap'_{i,1} + S_A \sum_{m=N_x}^{i+1} (p'_{m,1} - p'_{m,0})
\]

\[
A = \sqrt{\Phi} + \frac{\sigma_0 \Omega_0 \Delta x}{2c_0 \rho_0}, \quad G = \Omega_0, \quad S_A = \frac{\sigma_0 \Omega_0 \Delta x}{c_0 \rho_0}
\]

Consistency: two fluid layers

- Setting \( \sigma_0 = 0 \) and \( \Omega_0 = 1 \) gives:

\[
p'_{i,0} = \frac{\sqrt{\Phi} - 1}{\sqrt{\Phi} + 1} p'_{i,1} + \frac{2}{\sqrt{\Phi} + 1} p^g_{i,0} \quad \text{and} \quad p^g_{i,1} = \frac{1 - \sqrt{\Phi}}{\sqrt{\Phi} + 1} p^g_{i,0} + \frac{2\sqrt{\Phi}}{\sqrt{\Phi} + 1} p'^a_{i,1}
\]

\[\implies \text{two fluid layers with densities} \quad \rho_0 \text{ and} \quad \sqrt{\Phi} \rho_0\]
Two-way coupling equations:

\[
(A + G) p_{i,0}^{'a} = (A - G) p_{i,1}^{'a} + 2G p_{i,0}^{'g} + S_A \sum_{m=N_x}^{i+1} (p_{m,1}^{'a} - p_{m,0}^{'a})
\]

\[
(A + G) p_{i,1}^{'g} = (G - A) p_{i,0}^{'g} + 2A p_{i,1}^{'a} + S_A \sum_{m=N_x}^{i+1} (p_{m,1}^{'a} - p_{m,0}^{'a})
\]

\[
A = \sqrt{\Phi} + \frac{\sigma_0 \Omega_0 \Delta x}{2c_0 \rho_0}, \quad G = \Omega_0, \quad S_A = \frac{\sigma_0 \Omega_0 \Delta x}{c_0 \rho_0}
\]

Summary:

- First order limitations
- Causality
- Consistent with simple boundary conditions
Numerical implementation: FDTD

Linear terms: Crank-Nicolson method:
- Semi-implicit schemes
- Tridiagonal matrices

Nonlinearities: Flux corrected transport (FCT) algorithm:
- Explicit scheme
- Limit Gibbs oscillations

Two-way coupling:
- Only involves spatial derivatives and integrals
- No modifications to the solver
Propagation over an impedant ground surface

Source:
- Sine pulse, \( f = 1260 \text{ Hz} \)
- Position: \( x_s = 0, z_s = 1.4 \text{ m} \)

Receiver:
- Position: \( x_r = 10 \text{ m}, \) same height

3 ground layers:
- Rigid
- \( \sigma_0 = 5 \times 10^5 \text{ Pa.s.m}^{-2} \)
- \( \sigma_0 = 10^5 \text{ Pa.s.m}^{-2} \)
  with \( (\Phi = 3 \text{ and } \Omega_0 = 0.3) \)

Comparison to analytical calculations: solution to 2D Helmholtz equation
Rigid ground

\[ \sigma_0 = 5 \times 10^5 \text{Pa.s.m}^{-2} \]

\[ \sigma_0 = 10^5 \text{Pa.s.m}^{-2} \]
Reflected wave level and delay accurate for a wide range of ground characteristics
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High-amplitude propagation over porous layer:

- Additional absorption in the NPE model...

NPE model for porous ground layers

\[ D_t R = -\frac{c_0}{\sqrt{\Phi}} \frac{\partial}{\partial x} \left[ \left( 1 - \sqrt{\Phi} \right) R + \frac{\beta}{2} R^2 \right] \]

\[ -\frac{c_0}{2\sqrt{\Phi}} \int \frac{\partial^2 R}{\partial z^2} R dx - \frac{\sigma_0 \Omega_0}{2\Phi \rho_0} \left( 1 + \xi \frac{c_0}{\sqrt{\Phi}} |R| \right) R \]
High-amplitude propagation over porous layer:

- Additional absorption in the NPE model...
- ...but no modification of transmission and reflection coefficients

Two-way coupling equations:

\[
(A + G) p'_{i,0} = (A - G) p'_{i,1} + 2Gp'_{i,0} + S_A \sum_{m=N_x}^{i+1} \left( p'_{m,1} - p'_{m,0} \right)
\]

\[
(A + G) p'_{i,1} = (G - A) p'_{i,0} + 2Ap'_{i,1} + S_A \sum_{m=N_x}^{i+1} \left( p'_{m,1} - p'_{m,0} \right)
\]

\[
A = \sqrt{\Phi} + \frac{\sigma_0 \Omega_0 \Delta x}{2c_0 \rho_0}, \quad G = \Omega_0, \quad S_A = \frac{\sigma_0 \Omega_0 \Delta x}{c_0 \rho_0}
\]
Including Forchheimer’s non-linearities

Approximate solution

1. Artificially increase $\sigma$ in the coupling parameters with:

$$\sigma \rightarrow \sigma_0 \left(1 + \xi |w|\right)$$

2. Use the first-order approximation of $w$:

$$w = \left(\rho_0 c_0\right)^{-1} \int \partial_z p_1 \, dx + O\left(\varepsilon^{5/2}\right)$$

3. At the beginning of time step $n$, update the flow resistivity with values at time $n-1$:

$$\sigma^n \rightarrow \sigma_0 \left(1 + \frac{\xi}{\rho_0 c_0} \left|\int \partial_z p_1^{n-1} \, dx\right|\right)$$
Propagation with Forchheimer’s nonlinearities

Source:
- Sine pulse, $f = 700$ Hz
- Position: $x_s = 0$, $z_s = 3 \text{ m}$
- Peak amplitude: $3 \text{ kPa}$

Receiver:
- Position: $x_r = 10 \text{ m}$, same height

Soft layer:
- $\Phi = 3$
- $\Omega_0 = 0.3$
- $\sigma_0 = 10^4 \text{ Pa.s.m}^{-2}$

3 nonlinearity parameters:
- $\xi = 0 \text{ s.m}^{-1}$
- $\xi = 2.5 \text{ s.m}^{-1}$
- $\xi = 10 \text{ s.m}^{-1}$
\[ \xi = 0 \text{ s.m}^{-1} \]

\[ \xi = 2.5 \text{ s.m}^{-1} \]

\[ \xi = 10 \text{ s.m}^{-1} \]
Conclusions

- Accurate for a wide range of ground characteristics
- Simple implementation (only spatial integrals and derivatives)
- Forchheimer’s nonlinearities through approximate solution

Additions to the model

- Coupling equations for multi-layered ground surfaces can be derived
- NPE models (air & ground) and coupling equations can easily be adapted to handle non-flat topographies (terrain following coordinates)
Thank you for your attention

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