Nonlinear parabolic equation model for finite-amplitude sound propagation in an inhomogeneous medium over a non-flat, finite-impedance ground surface

T. Leissing\textsuperscript{a)}, P. Jean\textsuperscript{a)}, J. Defrance\textsuperscript{a)} and C. Soize\textsuperscript{b)}

\textsuperscript{a)} CSTB, 24 rue Joseph Fourier, 38400 Saint Martin d’Hères, France

\textsuperscript{b)} Université Paris-Est, Laboratoire Modélisation et Simulation Multi Echelle, MSME FRE3160 CNRS, 5 bd Descartes, 77454 Marne-la-Vallée, France

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Outline

1. Introduction
   - Context and objectives
   - Existing Nonlinear Parabolic Equation (NPE) models

2. NPE model with ground effects
   - NPE model for rigidly-framed porous ground layers
   - Interfacial boundary condition
   - Non-flat ground surfaces

3. Numerical validation
   - Propagation over an impedant ground surface
   - Propagation over a hilly ground

4. Concluding remarks
Shock waves from explosions can damage structures

Signals prediction at long range

→ Damages estimation

Propagation:

- Long range
- High amplitudes
- Meteorological effects
- Ground effects

**Nonlinear Parabolic Equation (NPE)** method well suited
Nonlinear outdoor sound propagation in complex environments

Introduction

Existing Nonlinear Parabolic Equation (NPE) models

Nonlinear Parabolic Equation (NPE)

\[ D_t R = -\partial_x \left[ c_1 R + c_0 \frac{\beta}{2} R^2 \right] - \frac{c_0}{2} \int \partial_z^2 R \, dx \]

with:

\[ R = \frac{\rho'}{\rho_0} \text{ and } \beta = \frac{\gamma + 1}{2} \]

Hypotheses:

1. Propagation along a main direction
2. Weak sound speed perturbations
3. Weak nonlinearities
Nonlinear outdoor sound propagation in complex environments

Introduction

Existing Nonlinear Parabolic Equation (NPE) models

Current models

- Various formulations (cylindrical, spherical, high-angle, ...)
- Sound speed perturbations, dissipation

Missing features

Ground effects: surface impedance, topography
Nonlinear outdoor sound propagation in complex environments
NPE model with ground effects

NPE $\rightarrow$ time-domain formulation

Complex impedance $Z(\omega)$ not usable $\rightarrow$ ground layer included in the computational system

TO DO:

1. Derive a NPE model for porous ground layers
2. Derive an interfacial boundary condition (ground/air coupling)
3. Adapt these formulations to non-flat surfaces
Extended Zwikker–Kosten model: 4 parameters:

1. Structure constant \((\Phi)\)
2. Porosity \((\Omega_0)\)
3. Flow resistivity \((\sigma_0)\)
4. Forcheimer’s nonlinearity parameter \((\xi)\)

Notes

- Sound speed: \(c_{\text{ground}} = \frac{c_0}{\sqrt{\Phi}}\)
- Nonlinear flow resistivity: \(\sigma = \sigma_0 + \xi |u|\)
- Impedance equivalence: \(Z(\omega) = \sqrt{\frac{\Phi}{\Omega_0^2}} + i \frac{\sigma_0}{\rho_0 \Omega_0 \omega}\)
Nonlinear outdoor sound propagation in complex environments

From Euler equations...
\[ \partial_t (\rho_T) + \nabla \cdot (\rho \mathbf{v}) = 0 \]
\[ \Phi \partial_t (\rho_T \mathbf{v}) + \Phi \nabla \cdot [\mathbf{v} \otimes (\rho \mathbf{v})] + \nabla (\rho_T) + \sigma_0 \Omega_0 (1 + \xi |\mathbf{v}|) \mathbf{v} = 0 \]

...to NPE model for porous ground layers
\[ D_t R = - \frac{c_0}{\sqrt{\Phi}} \partial_x \left[ \left( 1 - \sqrt{\Phi} \right) R + \frac{\beta}{2} R^2 \right] \]
\[ - \frac{c_0}{2\sqrt{\Phi}} \int \partial_x^2 R \, dx - \frac{\sigma_0 \Omega_0}{2 \Phi \rho_0} \left( 1 + \xi \frac{c_0}{\sqrt{\Phi}} |R| \right) R \]

- Reduced sound speed
- Frame speeds difference correction
- Nonlinear flow resistivity
Nonlinear outdoor sound propagation in complex environments

NPE model with ground effects

Interfacial boundary condition

**First-order interface condition**

1. Linearized vertical flow velocity equations
2. Moving frame operator: transforms \( t \)-integral to \( x \)-integral
   \[
   \left[ \sqrt{\Phi} \partial_z p^\text{air} - \frac{\sigma_0 \Omega_0}{\rho_0 c_0} \int \partial_z p^\text{air} \, dx \right] = \left[ \Omega_0 \partial_z p^\text{ground} \right]
   \]
3. Discretization \( \Rightarrow \) force values on corresponding grid points

**Note**

Does not take into account flow resistivity nonlinear part

\[ t = 7.1 \, ms \]
\[ \sigma_0 = 10 \, kPa.s.m^{-2} \]
Transformation: $z \rightarrow z' + h(x)$

$$\begin{align*}
\frac{\partial}{\partial x} &\rightarrow \frac{\partial}{\partial x} - h' \frac{\partial}{\partial z} \\
\frac{\partial^2}{\partial x^2} &\rightarrow \frac{\partial^2}{\partial x^2} + h'^2 \frac{\partial^2}{\partial z^2} - h'' \frac{\partial}{\partial z} - 2h' \frac{\partial}{\partial x} \frac{\partial}{\partial z}
\end{align*}$$

Nonlinear wave equation

$$\partial_t^2 \rho_T = \partial_x^2 (p_T + \rho_T u^2) + 2 \partial_x \partial_z (\rho_T uw) + \partial_z^2 (p_T + \rho_T w^2)$$
Nonlinear outdoor sound propagation in complex environments

- NPE model with ground effects
- Non-flat ground surfaces

Transformation: \( z \rightarrow z' + h(x) \)

\[
\begin{align*}
\partial_x & \rightarrow \partial_x - h' \partial_z \\
\partial_x^2 & \rightarrow \partial_x^2 + h'^2 \partial_z^2 - h'' \partial_z - 2h' \partial_x \partial_z
\end{align*}
\]

Generalized Terrain-NPE

\[
D_t R = -\partial_x \left[ c_1 R + c_0 \frac{\beta}{2} R^2 \right] - \frac{c_0}{2} \int \left[ (1 + h'^2) \partial_z^2 R + h'' \partial_z R \right] \, dx + c_0 h' \partial_z R
\]

- Gentle slopes
- Simple geometries
(Cannot handle screens, buildings, ... )
Numerical implementation: FDTD

Linear terms: Crank-Nicolson method:
- Semi-implicit schemes
- Tridiagonal matrices

Nonlinearities: Flux corrected transport (FCT) algorithm:
- Explicit scheme
- Limit Gibbs oscillations
Nonlinear outdoor sound propagation in complex environments

Numerical validation

Propagation over an impedant ground surface

Domain:
- 6 x 4 m
- Receiver: (6.0, 1.4) m

Source:
- Linear propagation
- Placed at x = 0, z = 1.4 m
- Gaussian spectrum: 600–1100 Hz

Ground layers: 3 cases:
- Rigid
  - \( \sigma_0 = 10^5 \text{ Pa.s.m}^{-2} \)
  - \( \sigma_0 = 10^4 \text{ Pa.s.m}^{-2} \)
  - with (\( \Phi = 3 \) and \( \Omega_0 = 0.3 \))

Comparison to BEM calculations
Rigid ground

\[ \sigma_0 = 10^5 \, Pa.s.m^{-2} \]

\[ \sigma_0 = 10^4 \, Pa.s.m^{-2} \]
Relative SPLs:

![Graphs showing relative SPLs for different ground characteristics.](image)

- Reflected wave level accurate for a wide range of ground characteristics.
Nonlinear outdoor sound propagation in complex environments

Numerical validation

Propagation over a hilly ground

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Configuration

Domain:
- 500 x 100 m

Hill:
- $h_{max} = 5 \text{ m at } x = 70 \text{ m}$
- $h'_{max} = 0.22$

Source:
- Linear propagation
- Placed at $x = 0, z = 50 \text{ m}$
- 50 Hz

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Comparison to BEM calculations
Nonlinear outdoor sound propagation in complex environments

Numerical validation

Propagation over a hilly ground

BEM

SPLs

NPE

Far field values close to BEM results
NPE model

With refraction and dissipation included, complete model for weakly nonlinear wave propagation

Propagation of waves from explosions: three-stage modelisation:

1. Near field: complete Euler equations
2. Moderate distances: NPE
3. Far-field (linear propagation): NPE or PE