Nonlinear parabolic equation model for finite-amplitude sound propagation over porous ground layers

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(Received 17 December 2008; revised 28 May 2009; accepted 3 June 2009)

The nonlinear parabolic equation (NPE) is a time-domain method widely used in underwater sound propagation applications. It allows simulation of weakly nonlinear sound propagation within an inhomogeneous medium. So that this method can be used for outdoor sound propagation applications it must account for the effects of an absorbing ground surface. The NPE being formulated in the time domain, complex impedances cannot be used and, hence, the ground layer is included in the computational system with the help of a second NPE based on the Zwikker–Kosten model. A two-way coupling between these two layers (air and ground) is required for the whole system to behave correctly. Coupling equations are derived from linearized Euler’s equations. In the frame of a parabolic model, this two-way coupling only involves spatial derivatives, making its numerical implementation straightforward. Several propagation examples, both linear or nonlinear, are then presented. The method is shown to give satisfactory results for a wide range of ground characteristics. Finally, the problem of including Forchheimer’s nonlinearities in the two-way coupling is addressed and an approximate solution is proposed.

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PACS number(s): 43.25.Cb, 43.25.Jh, 43.28.En, 43.28.Js [RR] Pages: 572–581

I. INTRODUCTION

Simulating outdoor shock wave propagation requires numerical models, which in addition to high-amplitude effects can account for meteorological conditions and ground properties. Indeed meteorological and ground effects can largely affect sound propagation and hence the wave temporal signature. These comprehensive propagation models are generally associated with an intense numerical effort. In this work a nonlinear parabolic equation (NPE) model is used to simulate finite-amplitude sound propagation. The NPE has first been developed by McDonald and Kuperman1 in 1987 and has been successfully used for underwater acoustics simulations.2 It has also been coupled to other numerical methods to simulate blast wave propagation in air.3–6 The NPE is based on the resolution of a nonlinear wave equation over a moving-window that surrounds the wavefront. While reducing domain size, and thus computational cost, the moving-window principle prevents backward propagation. For the derivation of the original NPE model, the reader may refer to articles by McDonald and co-workers7,8 or Caine and West.9 The NPE model for a two-dimensional (2D) domain with Cartesian coordinates \((x, z)\) is based on the following equation:

\[
D_t R = -\partial_i \left( c_1 R + c_0 \frac{\beta}{2} R^2 \right) - c_0 \frac{1}{2} \int \partial_j^2 R dx, \tag{1}
\]

where \(\partial_i\) means partial derivation with respect to variable \(i\), \(x\) is the main propagation direction, \(z\) is the transverse propagation direction, and \(t\) is the time variable. The ambient sound speed is \(c_0\) while \(c_1\) is the sound speed perturbation in the window, i.e., \(c_1 = c(x, z) - c_0\), where \(c(x, z)\) is the spatially-dependent sound speed. \(R = \rho' / \rho_0\) is a dimensionless over-density variable, with \(\rho'\) representing the acoustic density perturbation and \(\rho_0\) the ambient medium density. For air, the coefficient of nonlinearity \(\beta\) is calculated with the help of the ratio of specific heats \(\gamma\), i.e., \(\beta = (\gamma + 1)/2\). The first term on the right hand side of Eq. (1) represents refraction and non-linear effects; the second term accounts for propagation in the transverse direction. \(D_t\) is a moving-window operator and is defined by

\[
D_t = \partial_t + c_0 \partial_x. \tag{2}
\]

Note that in Eq. (1), the azimuthal spreading term \(c_1 R / (2r)\) has been dropped from the original NPE.1 The assumptions used to derive this model are (i) weak nonlinearities; (ii) weak sound speed perturbations, i.e., \(c_1 < c_0\); and (iii) propagation along a main direction. Equation (1) can be used to propagate weak shocks over moderate distances within a domain with spatially-varying sound speed. Various modifications and additions to this original model have been made during the past two decades: Spherical and cylindrical coordinate system versions10 and high-angle formulation11 have

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been developed, and Too and Lee\textsuperscript{12} extended the NPE with an additional term to account for thermoviscous effects. Propagation in multiple media\textsuperscript{13} and propagation through atmospheric turbulences\textsuperscript{14} were also successfully studied using this model.

Euler’s equations’ methods can provide complete solutions to nonlinear sound propagation problems:\textsuperscript{15} Realistic absorption models,\textsuperscript{16} meteorological effects,\textsuperscript{17,18} hilly terrain,\textsuperscript{19} and ground impedances\textsuperscript{20–23} can be accounted for in a very accurate way. Moreover, it does not suffer from the parabolic approximation inherent to NPE models. However, for long-range wave propagation problems, computational times will often be on the order of days for three-dimensional domains. Despite increasing computational resources and the existence of modern numerical techniques such as the use of efficient absorbing layers\textsuperscript{24} or adaptive mesh refinement methods,\textsuperscript{25} Euler’s equations-based models cannot compete in calculation time with NPE-based methods. Indeed the use of a single variable wave equation makes the NPE an efficient tool for long-range sound propagation simulations. The main motivation for the development of the NPE model presented here is its use to study finite-amplitude wave propagation over urban environments. Reduced computational times will allow this model to be used several hundred times to obtain statistical information on the wave fields.

In the present work, a parabolic equation model, which takes into account the effects of a soft ground layer on sound propagation, is proposed. It is not the objective here to study the propagation within the ground layer but rather to capture the effects of the non-rigid interface on the air acoustic fields. The paper is organized as follows. The derivation of the NPE model for porous ground layers is described in Sec. II. The combination of two-way coupling equations presented in Sec. III and two NPE models for atmospheric and porous ground media allows simulation of finite-amplitude sound propagation over an impedant ground surface. Several propagation examples are then shown in Sec. IV and, finally, an approximate solution to include Forchheimer’s nonlinearity\textsuperscript{26,27} in the two-way coupling is presented in Sec. V.

\section*{II. NPE MODEL FOR RIGIDLY-FRAMED POROUS MEDIA}

The domain considered is 2D with main axes $x$ (horizontal direction) and $z$ (vertical direction). Total density $\rho_T$ and total pressure $p_T$ variables are noted as follows:

$$\rho_T = \rho_0 + \rho', \quad p_T = p_0 + p',$$

where $\rho_0$ and $p_0$ are ambient air density and ambient air pressure, respectively, and $\rho'$ and $p'$ are acoustic perturbations of these quantities. Components of the flow velocity vector $V$ in the $x$- and $z$-directions are $u$ and $w$, respectively.

In order to preserve one of the most interesting features of NPE models, namely, a short simulation time, including the porous medium into the computational system, must not dramatically increase computational times. It is thus proposed to derive a parabolic equation model similar to Eq. (1), which uses a minimal parametrization: The layer is assumed to be equivalent to a continuous fluid medium. A wave causes a vibration of air particles contained in the ground pores, while the ground frame does not vibrate. The NPE model for sound propagation in porous ground media is based on a nonlinear extension of the Zwikker–Kosten (ZK) model,\textsuperscript{28} characterized by a set of four parameters: the dc flow resistivity $\sigma_0$, the porosity $\Omega_p$, the tortuosity $\Phi$, which is defined as the ratio of a curved path length to the distance between its end points, and the Forchheimer’s nonlinearity parameter $\xi$. These quantities are assumed to be fixed in space and time. Considering these assumptions equations of continuity and conservation of momentum are\textsuperscript{29–31}

\begin{align}
\partial_t \rho_T + \partial_x(\rho_T u) + \partial_z(\rho_T w) &= 0, \quad (4a) \\
\Phi \partial_t(\rho_T u) + \partial_x(\rho_T u^2) + \partial_z(\Phi \rho_T u w) + \sigma_0 \Omega_p [(1 + |\xi|)|u|] &= 0, \quad (4b) \\
\Phi \partial_t(\rho_T w) + \partial_x(\rho_T w^2) + \partial_z(\Phi \rho_T u w) + \sigma_0 \Omega_p [(1 + |\xi|)|w|] &= 0. \quad (4c)
\end{align}

As one can see in Eqs. (4a)–(4c), the tortuosity $\Phi$ reduces the pressure gradients and flow resistive terms’ amplitude. Combining Eqs. (4a)–(4c) gives

$$\Phi \partial_t^2 p_T = \partial_x^2(p_T + \Phi p_T u^2) + \partial_z^2(p_T + \Phi p_T w^2) + 2 \partial_x \partial_z (\Phi p_T u w) + \sigma_0 \Omega_p \partial_z [(1 + |\xi|)|u|] + \sigma_0 \Omega_p \partial_x [(1 + |\xi|)|w|].$$

Since the propagation is mainly along the $x$-axis, only linear terms in $z$-derivatives are kept in Eq. (5). Terms $\partial_x \partial_z (\Phi p_T u w)$, $\partial_z^2 (\Phi p_T u^2)$, and $\sigma_0 \Omega_p \partial_z [(1 + |\xi|)|u|]$ are neglected. Moreover, only terms of order up to two in $x$-derivatives are retained: The quantity $\partial_x^2(\Phi p_T u^2)$ is discarded; this leads to

$$\Phi \partial_t^2 p_T = \partial_x^2(p_T + \Phi p_T u^2) + \partial_z^2 p_T + \sigma_0 \Omega_p \partial_z [(1 + |\xi|)|u|] + \sigma_0 \Omega_p \partial_x [(1 + |\xi|)|w|].$$

(6)

To find an expression for the flow velocities $u$ and $w$ the authors use the perturbation expansion method. The same scalings and expansions as in Refs. 1 and 7 are used (however, note that the window speed in the ground layer is set to $c_0/\sqrt{\Phi}$) as follows:

$$x \rightarrow x - \frac{c_0}{\sqrt{\Phi}} t, \quad z \rightarrow \epsilon^{1/2} z, \quad t \rightarrow \epsilon t.$$

(7)

The scaling of $z$ by a factor of $\epsilon^{1/2}$ emphasizes the predominance of the propagation in the $x$-direction. The partial derivatives associated with Eq. (7) are

$$\partial_x \rightarrow \partial_x, \quad \partial_z \rightarrow \epsilon^{1/2} \partial_z, \quad \partial_t \rightarrow \epsilon \partial_t - \frac{c_0}{\sqrt{\Phi}} \partial_x.$$ 

(8)

The dependent variables are expanded as follows:

$$p_T \rightarrow p_0 + \epsilon p_1 + \epsilon^2 p_2 + \epsilon^3 p_3 + \cdots,$$

$$u \rightarrow \epsilon u_1 + \epsilon^{3/2} u_2 + \epsilon^2 u_3 + \cdots.$$ 

(9)

\textsuperscript{29} J. Acoust. Soc. Am., Vol. 126, No. 2, August 2009 Leissing et al.: Shock wave propagation over porous grounds
Inserting Eqs. (8) and (9a)–(9c) in Eq. (4a) gives
\[
\left( \epsilon \partial_t - \frac{c_0}{\sqrt{\Phi}} \partial_x \right) (p_0 + \epsilon p_1 + \epsilon^2 p_2 + \cdots) = -\partial_x \left[ (p_0 + \epsilon p_1 + \epsilon^2 p_2 + \cdots) \right] - \epsilon^2 \partial_x^2 \left[ (p_0 + \epsilon p_1 + \epsilon^2 p_2 + \cdots) \right].
\]
(10)

A first-order approximation of the flow velocity components can be obtained by equating terms of orders \( \epsilon \) and \( \epsilon^{3/2} \) in Eq. (10) as follows:
\[
u_1 = \frac{c_0}{\sqrt{\Phi}} \rho_1, \quad w_1 = 0.
\]
(11)

Note that \( \rho' = \rho_1 + O(\epsilon^2) \), \( u = u_1 + O(\epsilon^{3/2}) \), and \( w = w_1 + O(\epsilon^{3/2}) \). The substitution of \( u \) and \( w \) by \( u_1 \) and \( w_1 \) in Eq. (6) leads to an error consistent with the accuracy sought. One obtains
\[
\Phi \partial_t \rho_T = \partial_x^2 \left( \rho_T + c_0^2 \frac{\rho_T^{1/2}}{\rho_0} \right) + \partial_x^2 \rho_T + \frac{\sigma_0 \Omega_0 \rho_0}{\rho_0 \sqrt{\Phi}} \partial_x \left( 1 + \frac{\xi c_0}{\sqrt{\Phi}} \rho' \right). \tag{12}
\]

In order to reduce Eq. (12) to a single variable equation, the total pressure \( \rho_T \) is substituted by a second-order expansion in \( \rho' \) from an assumed adiabatic equation of state
\[
\rho_T = \rho_0 + c_0^2 \rho' + c_0^2 \left( \frac{\gamma - 1}{2 \rho_0} \right) \rho'^2,
\]
where \( \gamma \) is the ratio of specific heats. Inserting Eq. (13) in Eq. (12) yields
\[
\Phi \partial_t \rho' = \left[ \rho' + \left( \frac{\gamma + 1}{2 \rho_0} \right) \rho'^2 \right] + c_0^2 \partial_x^2 \rho' + \frac{\sigma_0 \Omega_0 \rho_0}{\rho_0 \sqrt{\Phi}} \partial_x \left( 1 + \frac{\xi c_0}{\sqrt{\Phi}} \rho' \right). \tag{14}
\]

The one-way propagation hypothesis is introduced with a moving-frame operator \( D^*_i \) as follows:
\[
D^*_i = \partial_i + \frac{c_0}{\sqrt{\Phi}} \partial_x. \tag{15}
\]

The first-order parabolic approximation gives \(^9\)
\[
\partial_t^2 \sim -2 \frac{c_0}{\sqrt{\Phi}} D^*_i \partial_x + \frac{c_0^2}{\Phi} \partial_x. \tag{16}
\]

Replacing the second time derivative in Eq. (14) gives a NPE model for propagation in a porous medium
\[
D^*_i R = -\frac{c_0}{\sqrt{\Phi}} \partial_x \left( \frac{\beta R^2}{2} \right) - \frac{\sigma_0 \Omega_0}{2 \Phi \rho_0} \partial_x \left( 1 + \frac{\xi c_0}{\Phi} \rho' \right) R. \tag{17}
\]

Equation (17) can be used to simulate sound propagation within a porous ground layer. However, if one wants to couple air/ground models, a last modification must be done. Indeed, both models use different moving-window speeds: \( c_0 \) and \( c_0 / \sqrt{\Phi} \). Correcting for the frame-speed difference leads to the following substitution:
\[
D^*_i \rightarrow D_i + \frac{c_0}{\sqrt{\Phi}} (1 - \sqrt{\Phi}) \partial_x. \tag{18}
\]

Equation (17) becomes
\[
D_i R = -\frac{c_0}{\sqrt{\Phi}} \partial_x \left( 1 - \sqrt{\Phi} R + \frac{\beta R^2}{2} \right) - \frac{c_0}{2 \Phi} \partial_x^2 R dx - \frac{\sigma_0 \Omega_0}{2 \Phi \rho_0} \left( 1 + \frac{\xi c_0}{\Phi} R \right) R. \tag{19}
\]

The NPE model described by Eq. (19) is able to simulate finite-amplitude sound propagation within a rigidly-framed porous material described by a set of four parameters. Note that if one sets \( \Phi = 1 \) and neglects losses in the layer, i.e., \( \sigma_0 = 0 \), the model exactly reduces to the usual NPE model for atmospheric propagation given in Eq. (1). Equation (19) allows to draw some conclusions about finite-amplitude sound propagation in porous media: (i) The sound speed in the medium is inversely proportional to the square root of the material tortuosity, i.e., \( c = c_0 / \sqrt{\Phi} \); (ii) the attenuation in the ground layer is composed of a linear term plus a nonlinear term; and (iii) with the hypothesis used, the material resistivity is proportional to the overdensity \( R \).

### III. DERIVATION OF TWO-WAY COUPLING EQUATIONS

As NPE models for air and ground layers use the same moving-frame speed, they can be combined to simulate finite-amplitude sound propagation over a rigidly-framed porous ground layer. This section aims at establishing first-order coupling equations to link these two parabolic propagation models. In the following the authors assume that the deformation of the interface by the wave is small.\(^ {13} \)

#### A. Derivation

An air layer, whose fields are noted \( p^{\infty}, u^{\infty}, \) and \( w^{\infty} \), is considered. To construct the air-ground interfacial condition a rigidly-framed porous ground layer is introduced; its fields are noted \( p^{\infty}, u^{\infty}, \) and \( w^{\infty} \). With these notations interfacial boundary conditions are continuity of pressure and normal flow velocity
\[
[p^{\infty}] = [p^{\infty}], \quad [w^{\infty}] = [w^{\infty}], \tag{20}
\]
where the square brackets denote the field quantity on the air-ground interface. Expressions of \( w^{\infty} \) and \( w^{\infty} \) involving the pressure disturbance \( p' \) to the first order are sought; linearized equations are hence used. For the air layer the authors use the linearized Euler’s equation
\[
\rho_0 \partial_t (w^{\infty}) = -\partial_t p^{(\infty)}. \tag{21}
\]

The perturbation expansion method is used and the same scalings as in Sec. II and in Refs. 1 and 7 are used. Rewriting Eq. (21) and equating terms of orders 1 and 3/2 give
Rearranging Eq. sought in this work it can be written as

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Replacing these approximations with Eq. virtual points with pressures

\[ p_{i,j}^a, p_{i,j}^g \]

For a generic layer

\[ p_{A}^{\text{trapezoidal law}}, \frac{\partial p_{i,j}^a}{\partial x} \]

and their derivatives are used to discretize Eq.

\[ \frac{\partial p_{i,j}^a}{\partial x} \]

where \( N_c \) is the number of points in the moving-window in the x-direction and

\[ A = \sqrt{\Phi} + \frac{1}{2} S, \]

\[ G = \Omega_0, \]

\[ S = \sigma_0 \Omega_0 \Delta x \]

Equations (29) and (30) give expressions for the unknown pressures \( p_{i,0}^a \) and \( p_{i,1}^g \) and thus allow, used together with the atmospheric and porous ground NPE models, to simulate weakly nonlinear sound propagation over an impedant ground.

### C. Properties

In this section, fundamental properties of the boundary conditions are described and some notes about its numerical implementation are given.

**Limitations.** First-order formulations of the constitutive equations have been used to derive the boundary interface condition. This implies that nonlinearities cannot be taken into account in the two-way coupling.

**Causality.** The x-integral present in NPE models [see, for example, Eq. (1)] is calculated from right to left in the calculation grid, and the same method is used for coupling [note the reversed sum indices in Eqs. (29a) and (29b)]. This ensures that no perturbation is introduced ahead of the point where the wave hits the ground, and thus implies that the interfacial condition is causal.

**Consistency with classical boundary conditions.** If one sets \( \Phi = +\infty \) one obtains the following from Eqs. (29a) and (29b): \( p_{i,0}^a = p_{i,1}^g \), which, with the discretization used, the condition for a totally rigid interface. A transparent interface condition can be obtained by setting \( \alpha_0 = 0, \Omega_0 = 1, \Phi = 1 \) (parameters for an air layer). This leads to \( A = 1 \) and \( G = 1 \) and thus \( p_{i,0}^a = p_{i,1}^g \) and \( p_{i,0}^g = p_{i,1}^g \), which is the condition for perfect transmission. If one sets \( \alpha_0 = 0 \) and \( \Omega_0 = 1 \), Eqs. (29a) and (29b) become

\[ p_{i,0}^a = \frac{\sqrt{\Phi} - 1}{\sqrt{\Phi + 1}} p_{i,0}^g + \frac{2}{\sqrt{\Phi + 1}} p_{i,1}^g, \]

\[ p_{i,1}^g = (A - G) p_{i,0}^g + 2G p_{i,0}^g + S \sum_{m=N_c}^{i+1} (p_{m,1}^{\text{tr}} - p_{m,0}^{\text{tr}}), \]

\[ (A + G) p_{i,1}^g = (A - G) p_{i,0}^g + 2A p_{i,0}^g + S \sum_{m=N_c}^{i+1} (p_{m,1}^{\text{tr}} - p_{m,0}^{\text{tr}}), \]
which is the interface condition for two fluid layers with densities $\rho_0$ and $\sqrt{\Phi}\rho_0$.

**Numerical implementation.** To solve the diffraction operator a first-order finite-difference approximation for spatial discretization and the Crank–Nicolson method for time marching are used. This leads to a tridiagonal system of equations that is solved columnwise, from right to left in the calculation grid. The boundary interface condition can thus be naturally included in the diffraction solver by imposing values on corresponding points without any additional solver modifications.

**IV. NUMERICAL EXAMPLES**

In this section, numerical examples of sound propagation over porous ground layers are presented to illustrate the coupling method and to evaluate its performances.

**A. Linear propagation**

**1. Reference solutions**

The solutions of the 2D Helmholtz equation are used as references. In the case of 2D wave propagation over a flat and impenetrable ground surface the pressure is given by

$$p_r = \frac{1 - \sqrt{\Phi}}{\sqrt{\Phi} + 1} p_i + \frac{2\sqrt{\Phi}}{\sqrt{\Phi} + 1} p_i,$$

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\( =500 \text{ kPa s m}^{-2} \text{ and } \sigma_0 = 100 \text{ kPa s m}^{-2} \). The NPE window including the ground layer is 3 m wide and 4.125 m high (400 \times 550 points).

### 3. Results

Two modifications are done on signals at the receiver: First, in order to obtain a free field reference, time histories are cropped after the direct wave. Next the trailing part of time signals is cropped after the reflected wave to suppress the low-amplitude numerical oscillations. Let us denote the complex pressures at the receiver and in the free field by \( p_r \) and \( p_{\text{free}} \) respectively. The relative sound pressure level (SPL) \( \Delta L \) is then calculated with

\[
\Delta L = 10 \log \left( \frac{p_r}{p_{\text{free}}} \right).
\]

Note that Salomons et al.\(^ {20} \) found very little difference on relative SPLs when comparing sources with different decay rates. Analytical solutions for cylindrical line sources are used in this work.

Relative SPLs at the receiver are shown in Fig. 2, for both analytical and NPE calculations. Very good agreement can be observed, independently of the ground properties: Even for the softest layer (\( \sigma_0 = 100 \text{ kPa s m}^{-2} \)) the difference between analytical and NPE calculations is at most 1 dB. The frequencies where negative interference occurs are 1325, 1273, and 1246 Hz for the rigid case, the ground layer with \( \sigma_0 = 500 \text{ kPa s m}^{-2} \), and the ground layer with \( \sigma_0 = 100 \text{ kPa s m}^{-2} \), respectively. As one can see in Fig. 2 the NPE model presented does not accurately recreate reflected wave amplitude decrease, but does account for the change in least reflective frequencies due to the additional delay given during reflection.

### B. Nonlinear propagation

#### 1. Reference solution

To obtain reference results to compare to the NPE model simulations in the case of high-amplitude waves, solutions of Euler’s equations are used. The computational domain is composed of an air layer and a ground layer. In a 2D Cartesian coordinate system the constitutive equations for the air layer are

\[
\partial_t \rho_T + \partial_x (\rho_T u) + \partial_z (\rho_T w) = 0, \tag{35a}
\]

\[
\partial_t (\rho_T u) + \partial_x (\rho_T u^2) + \partial_z (\rho_T u w) = - \partial_x p_T, \tag{35b}
\]

\[
\partial_t (\rho_T w) + \partial_x (\rho_T u w) + \partial_z (\rho_T w^2) = - \partial_x p_T, \tag{35c}
\]

\[
\partial_t (\rho_T e_0) + \partial_x (\rho_T u e_0) + \partial_z (\rho_T w e_0) = - \partial_x (p_T u) - \partial_z (p_T w), \tag{35d}
\]

where \( e_0 \) is the energy per unit mass. Within the ground layer momentum conservation equations write

\[
\Phi \partial_t (\rho_T u) + \partial_x (\rho_T u^2) + \partial_z (\Phi \rho_T u w) + \sigma_0 \Omega_0 (1 + \xi[u]) u = 0, \tag{36a}
\]

\[
\Phi \partial_t (\rho_T w) + \partial_x (\rho_T u w) + \partial_z (\Phi \rho_T w^2) + \sigma_0 \Omega_0 (1 + \xi[w]) w = 0. \tag{36b}
\]

The energy equation (37) and the ideal gas law (38) close the equation system

\[
\rho_T e_0 = \rho_T C_v T + \frac{\rho_T |V|^2}{2}, \tag{37}
\]

\[
\rho_T = \rho_T R T, \tag{38}
\]

where \( T \) is the gas temperature, \( C_v \) is the specific heat capacity at constant volume, and \( R \) is the gas constant. To solve this equation system a weighted essentially non-oscillatory (WENO) algorithm\(^ {33} \) for space discretization and a third-order total variation diminishing scheme\(^ {34} \) for time marching are used. These numerical algorithms are briefly presented in the Appendix.
2. Configuration

In this example, standard atmospheric conditions are used \((T=293\ K, \rho_0=1.2\ \text{kg m}^{-3}, \text{and} \ p_0=1.03 \times 10^5\ \text{Pa})\). The source is positioned at \((x_r,z_r)=(0,3)\) m and the receiver at \((x_r,z_r)=(12,3)\) m.

In order to start the reference calculation, the pressure, velocity, density, and energy fields need to be specified. A Gaussian pulse is propagated using a one-dimensional version of the code presented in the Appendix. By adjusting the pulse amplitude and width one can obtain a one-dimensional signal at a given distance. In this example an amplitude and signal length of approximately 4 kPa and 1.5 m, respectively, were aimed for at a distance of 3 m from the source. Spatial steps are equal to 0.015 m in both directions, leading to a resolution of approximately 100 points per wavelength. This signal is then spherically extrapolated to obtain a 2D array. Figure 3 shows the one-dimensional signal and its 2D extension used to start both reference and NPE calculations.

A simulation on a perfectly rigid ground has been performed together with two calculations on different ground layers. Both have identical tortuosity \((\Phi=3)\) and porosity \((\Omega_0=0.3)\) but have different flow resistivity values \((\sigma_0=100\ \text{kPa s m}^{-2}\) and \(\sigma_0=10\ \text{kPa s m}^{-2}\)). These flow resistivity values have been chosen to test the model limitations rather than to represent a real situation. Chosen flow resistivities would correspond to grass \((\sigma_0=100\ \text{kPa s m}^{-2}\) and light, dry snow \((\sigma_0=10\ \text{kPa s m}^{-2}\). The ground layer is 75 cm thick (50 points) and for NPE calculations the moving-window is 4.5 m wide and 6 m high \((300 \times 400)\) points).

3. Results

Figure 4 shows snapshots of the propagation for non-rigid ground layers at time \(t=33\) ms for both models. Colormaps represent results from the NPE model while contour lines are results from Euler’s equations. Time signals are recorded at the receivers; Fig. 5 shows these signals for NPE and reference calculations for the three ground layers considered. Although Euler’s equations’ model seems to smear out reflected waves more than the NPE model, the parabolic propagation model produces time waveforms comparable to the reference ones.

To evaluate the accuracy of the NPE model, some characteristics of the reflected wave are studied: the maximum positive and negative peak pressures and their arrival times (noted, respectively, \(p_+\) and \(p_-\), and \(t_+\) and \(t_-\)), and the positive phase duration (noted \(t_d\)). These characteristics are summarized in Table I. Since for the softest ground layer the negative peak on the reflected wave is very weak, values of \(p_-\) and \(t_-\) for this layer are irrelevant.

As one can see, arrival times differ by at most 0.3 ms. The difference is larger for the softest layer; a possible reason is that the NPE model does not smear out pulses as the reference model does, leading to erroneous positive peak position. One can thus expect that as the flow resistivity decreases the error on arrival time increases. However, in outdoor sound propagation applications, the flow resistivity may seldom be lower than the one used here \((\sigma_0=10\ \text{kPa s m}^{-2}\), so the error on arrival time will remain weak for most cases. These remarks are also applicable to the positive phase duration \(t_d\). Positive peak amplitudes differ by 6.2% and 5.1% for layers with \(\sigma_0=100\ \text{kPa s m}^{-2}\) and \(\sigma_0=10\ \text{kPa s m}^{-2}\), respectively. This difference does not seem to depend on flow resistivity and, as a comparison, the relative error for the perfectly rigid layer is 1%. Relative errors for negative peaks are comparable: 2.4% and 3.4% for the rigid layer and the layer with \(\sigma_0=100\ \text{kPa s m}^{-2}\), respectively.

As a mean of comparison, calculation times for Euler and NPE models were about 3.5 h and 4 min, respectively (calculations were done on a modern desktop computer). Although Euler’s equations’ implementation could use more advanced numerical techniques (adaptive mesh refinement methods\(^{25}\) and moving-window principle\(^{15}\), the NPE model, thanks to the use of a single variable one-way wave equation and a fast solver (Thomas algorithm), is a very efficient tool for outdoor sound propagation simulations.

V. INCLUDING FORCHEIMER’S NONLINEARITIES IN THE TWO-WAY COUPLING

While the flow resistivity dependence on particle velocity (Forchheimer’s nonlinearities) is accounted for in the NPE model for porous ground layers [last term in Eq. (19)], the two-way coupling between both domains does not contain high-amplitude effects on ground properties. This would lead to erroneous solutions, since an additional attenuation would be introduced in the ground layer, but the increased rigidity of the interface would not be accounted for.

A solution is to artificially increase the flow resistivity in Eqs. (30a)–(30c) according to

\[\sigma_0 \rightarrow \sigma_0 + \delta \sigma_0,\]

where \(\delta \sigma_0\) is a small positive constant that can be adjusted to obtain the desired level of attenuation. This approach is commonly used in outdoor sound propagation models to account for the increased rigidity of the interface.
The relative errors for positive and negative peak pressures are 4.38% and 4.34%, respectively, while the error on positive phase duration and time of arrival of positive peak pressure is 0.7 ms and 0.6 ms for Euler and NPE simulations, respectively, while the positive phase duration is reduced by 0.3 ms for both models. The signals modifications due to the addition of Forchheimer’s nonlinearities are nearly identical for both models, confirming that the method presented to take into account the flow resistivity dependence on particle velocity is accurate.

### VI. CONCLUSION AND PERSPECTIVES

A NPE model based on a nonlinear extension of the ZK model has been developed; it allows simulation of weakly nonlinear propagation within a porous ground layer. Next, two-way coupling equations have been derived from linearized Euler’s equations. This interfacial boundary condition couples air and ground NPE models and enables the NPE to account for the effects of soft ground layers on sound propagation. For linear propagation, the results obtained with this method have shown very good agreement with analytical solutions for a wide range of ground properties. For high-

![Fig. 6. Time signals at the receiver for Forchheimer’s parameter $\xi = 2.5 \text{ s m}^{-1}$. Solid line: Euler; dotted line: NPE.](image-url)
amplitude waves, the NPE model produces time signals comparable to those obtained by Euler’s equations’ model. Relative error on peak pressures has been shown to be independent of material properties while differences on positive phase duration and time of arrival increase with decreasing ground flow resistivity. However, the presented model still gives good agreement even for very low flow resistivities and provides a simple but efficient way of taking into account ground impedances. Finally, an approximate method to include Forchheimer’s nonlinearities in the two-way coupling has been presented: It consists in artificially increasing the flow resistivity value in the coupling parameters. This method has been proven to give satisfactory results and does not introduce any additional source of error in the two-way coupling.

To construct the NPE model, the assumption that the ground layer is equivalent to a continuous fluid has been made. This simplified approach allows derivation of a ground model that is of the same form as the NPE model for atmospheric layer. Since the two-way coupling equations involve only spatial derivatives and integrals, the complete NPE model is able to perform simulations in a very short time (about 50 times faster than Euler’s equations’ implementation). This enables the NPE model to be used as a stochastic model solved by the Monte Carlo method. Wave field statistics in the air layer could be determined by performing a large number of simulations of sound propagation in an environment with varying parameters (e.g., propagation over a ground layer with random flow resistivity and propagation through turbulences). However, note that for realistic simulations, a spherical spreading term should be added to the NPE used in this work.

The relative simplicity of the NPE model and its coupling method makes it a good candidate for extensions and modifications. In a previous work, the NPE models for porous ground layers and two-way coupling equations have been adapted to handle non-flat topographies, through the terrain-following coordinates method. Two-way coupling equations could also be derived for multilayered ground surfaces without much additional work. With atmospheric refraction and dissipation included, it would provide a complete NPE model for weakly nonlinear wave propagation including most of the features of sound propagation outdoors (refraction, dissipation, topography, and ground impedance effects). This tool could be used, for instance, for propagating waves from explosions using a three stage procedure: first, a method based on Euler’s equations could be used in the near field, where the propagation is highly nonlinear. Next, NPE models could propagate weakly nonlinear waves over moderate distances and, finally, when the wave amplitude is low enough, frequency-domain method such as the parabolic equation (PE) could be used. This hybrid method allows propagation of waves from explosions over distances up to several kilometers.

<table>
<thead>
<tr>
<th>Model</th>
<th>( t_{e+} ) (ms)</th>
<th>( p_{e+} ) (Pa)</th>
<th>( t_{d} ) (ms)</th>
<th>( t_{e-} ) (ms)</th>
<th>( p_{e-} ) (Pa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euler</td>
<td>38.6</td>
<td>246</td>
<td>2.1</td>
<td>40.4</td>
<td>115</td>
</tr>
<tr>
<td>NPE</td>
<td>38.4</td>
<td>258</td>
<td>1.7</td>
<td>40.5</td>
<td>120</td>
</tr>
</tbody>
</table>

**TABLE II.** Reflected wave characteristics for reference and NPE calculations with Forchheimer’s nonlinearities.

<table>
<thead>
<tr>
<th>Model</th>
<th>( t_{e+} ) (ms)</th>
<th>( p_{e+} ) (Pa)</th>
<th>( t_{d} ) (ms)</th>
<th>( t_{e-} ) (ms)</th>
<th>( p_{e-} ) (Pa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euler</td>
<td>–0.7</td>
<td>+21.78%</td>
<td>–0.3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>NPE</td>
<td>–0.6</td>
<td>+21.12%</td>
<td>–0.3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**TABLE III.** Differences in reflected waves characteristics with and without Forchheimer’s nonlinearities. Results are shown for both NPE and reference calculations.

\[ w^{n+1} = \frac{1}{3} w^n + \frac{2}{3} w^{(2)} + \frac{2}{3} \Delta t K^{(2)}, \]

where, for the air layer, \( w^n \) is the solution vector at time iteration \( n \), i.e.,

\[ w^n = \begin{pmatrix} \rho_T \\ \rho_T u \\ \rho_T w \\ \rho_T v \end{pmatrix} \]

and \( K^{(i)} \) is the right hand side of the equation system, i.e.,

ACKNOWLEDGMENTS

The authors would like to thank F. Van Der Eerden and E. Salomons for providing the initialization array and answering their questions about it.

APPENDIX: NUMERICAL SOLUTION OF EULER’S EQUATIONS

The main principle of the WENO scheme is the use of multiple stencils to evaluate the derivative at a given point. The algorithm first determines where there is a discontinuity in the solution, and then weights stencils accordingly to avoid spurious numerical oscillations. These features make the WENO scheme accurate for propagating shock waves. For the sake of brevity, computation details are omitted here, but the reader may refer to the work of Shu or Wochner.

The time discretization scheme is of the form

\[ w^{(1)} = w^n + \Delta t K^n, \]
\[ w^{(2)} = \frac{3}{4} w^n + \frac{1}{4} w^{(1)} + \frac{1}{4} \Delta t K^{(1)}, \]
\[ w^{n+1} = \frac{1}{3} w^n + \frac{2}{3} w^{(2)} + \frac{2}{3} \Delta t K^{(2)}, \]
Note that for the ground layer, $\mathbf{w}$ and $\mathbf{K}^{(0)}$ have to be modified according to Eqs. (36a) and (36b).

Although the combination of WENO and Runge–Kutta schemes allow to stably propagate discontinuities, it is not able to propagate waves of infinite slope: A shock smearing will occur where the slope is too steep, resulting in small deviations from physical solutions for very high-amplitude waves.

$\mathbf{K}^{(i)} = -\partial_z \left( \begin{array}{c} \rho_T h^2 \\ \rho_T h w \\ \rho_T e_\theta u \\ 0 \\ \partial_z p_T \\ -\partial_z (p_T u) - \partial_z (p_T w) \end{array} \right) \right) (i).$ (A3)